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Oscillatory Behavior of Solution of Hilfer Fractional Differential Equation

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ABSTRACT: In this paper, we study the oscillation of impulsive fractional differential equations.Using the inequality principle and Bihari Lemma,sufficient conditions are found for both the asymptotic and oscillatory phases of the equation. An example is given to illustrate the validity of our main results. The oscillation of an impulsive fractional differential equation with two different Caputo derivatives is being studied for the first time.

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I. INTRODUCTION

Fractional differential equations are becoming more common in various research fields, essential for describing the mechanical and electrical behaviors of physical materials. They find application in rheological theory and various other physical phenomena, see[1–3]. To explore details regarding the oscillation of fractional differential equations, readers are encouraged to refer to the pertinent literature[4–9]. Lately, several authors have investigated the oscillatory tendencies exhibited by various categories of fractional differential equations and impulsive partial differential equations[10–15,17–19]. Fractional differential equation and inclusions involving Caputo derivative or Riemann-Liouville derivative have obtained more and more results see[20–28]. Recently, Hilfer initiated an extended Riemann-Liouville fractional derivative, named Hilfer fractional derivative, which interpolates Caputo fractional derivative and Riemann Liouville fractional.This operator emerged in theoretical simulations of dielectric relaxation in glass-forming materials. Hilfer et al.[29] introduced linear differential equations using the Hilfer fractional derivative and utilized operational calculus to solve these generalized fractional differential equations.

In[10,15,19], the authors have formulated conditions that adequately describe the oscillatory tendencies observed in fractional differential equations that include damping terms. In[11], the authors have investigated the oscillatory features of solutions in a nonlinear fractional partial differential equation containing damping and a forced term, specifically under Robin boundary conditions. Furthermore, the authors have investigated recent progress in understanding the oscillatory properties of solutions in fractional ordinary differential equations[14,16]. In[18] Li has identified sufficient conditions for forced oscillations in specific partial fractional differential equations using methods that incorporate differential inequalities. In[30] the researchers explored the controllability and optimal control aspects of a system characterized by two distinct fractional orders, $0<\alpha<\beta<1$

$$\begin{cases} D^{\beta}_{0+z} z(u) &= Az(u)+Bz(u-\tau)+C\cdot D^{\alpha}_{0+}z(u)+Gz(u), \quad u\geq 0 \\ z(u) &= \phi(u); -\tau\leq u\leq 0 \end{cases}$$

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In[10], the researchers analyzed the oscillatory dynamics of an impulsive fractional differential equation featuring a Caputo derivative in a specific form

$$\begin{cases} {}^{c}D_0^q z(u) = e(u) + f(u, z(u)), \quad b > 1, \quad u \in I' := I/u_1, \ldots, u_m, \quad I := [b, \infty,) \\ \triangle z(u_{r_1}) = y_{r_1}, \triangle z'(u_{r_1}) = y^- r_1, \quad r_1 = 1, 2, \ldots\ldots \\ z(b) = z(0), z'(b) = \bar{z} \end{cases}$$

where ${}^{c}D_0^q$ is the Caputo derivative of the order $q \in (1,2), z_0, \bar{z}, y_{r_1}, y^- r_1 \in \mathbb{R}, u_{r_1}$ satisfy $0 = u_0 < \ldots < u_m \to \infty$ as $m \to \infty . \Delta v(u_{r_1}) = v(u_{r_1}^+) - v(u_{r_1}^-)$ with $v(u_{r_1}^+) = \lim_{\epsilon \to \infty^+}, v(u_{r_1} + \epsilon)$ and $v(u_{r_1}^-) = \lim_{\epsilon \to \infty^-}, v(u_{r_1} + \epsilon)$ represent the right and left limits of $v(u)$ at $u = u_{r_1}$. Inspired by previous research. We concern the oscillatory behavior of solutions in a fractional impulsive Hilfer differential equation

$$\begin{cases} {}^{H}D_b^{p,q} z(u) = \lambda e(u) + f(u, z(u)) + \sum_{i=1}^{n} b_i(u) g_i(z(u - \tau_i)), \quad u \in I' := I/u_1, \ldots, u_m, \\ I := [b, \infty), \\ \left(I_b^{1-r} \triangle z\right)(u_{r_1}) = y_r, \quad r = 1, 2, \ldots \\ I_b^{1-r} z(b) = z_0, \end{cases} \tag{1}$$

where ${}^{H}D_b^{p,q}$ is the Hilfer fractional derivative of the order $0 \le q < p \le 1$ and $r = p + q - pq$, z_0, $\bar{z}$, y_{r_1}, $\bar{y}_{r_1} \in \mathbb{R}$, $u + r_1$ satisfy $b = u_0 < \ldots < u_m \to \infty$ as $m \to \infty . \Delta v(u_{r_1}) = v\binom{+}{r_1} - v(u_{r_1}^-)$ with $v(u_{r_1}^+) = \lim_{\epsilon \to 0^+} v(u_{r_1} + \epsilon)$ and $v(u_{r_1}^-) = \lim_{\epsilon \to 0^-} v(u_{r_1} + \epsilon)$ represent the right and left limits of $v(U)$ at $u = u_{r_1}$. The objective of this paper is to explore the oscillatory dynamics of Hilfer fractional differential equations. The oscillatory sufficient condition for (1) is derived through an analysis of its integral expression. To facilitate the initial investigation of fractional differential equation oscillation with Hilfer derivative, we employ an exceedingly simple impulsive condition.

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$$\left(I_b^{1-r} \triangle z\right)(u_{r_1}) = y_{r_1}, \quad r = 1, 2, \ldots\ldots$$

II. MAIN RESULTS

We are now in a position to state and prove our main results.

Theorem 2.1 If $0 < p < 1, \quad 0 \le q \le 1, \quad \mu > 0, \quad \alpha > 1, \beta = \frac{\alpha}{\mu - 1}, \alpha(p-2) + 1 > 0$, and the function $z(u) : I \to \mathbb{R}$ is continuous, then

$$\frac{\lambda}{u} \int_0^u (u - w)^{p-1} |e(w)| dw \quad \text{is bounded for all} \quad u \ge 0, \tag{2}$$

as well as the function $f(u, z(u))$ satisfy the given conditions. (i) In $D = (u, z(u)) : u \in J, z \in \mathbb{R}, f(u, z(u))$ is continuous. (ii) Non negative continuous functions g and h are defined on $\mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$, with g being non decreasing and let $0 < \mu \le 2 - p - 1/\alpha$ such that

$$|f(u,z)| \le u^{\mu-1}h(t)g\left(\frac{|z|}{u}\right), \quad u>0, \quad (u,z)\in D \quad (3)$$

and

$$\int_0^{infty} w^{\beta\theta/\alpha}h^{\beta}(w)dw < \infty \quad (4)$$

holds, where $\theta := \alpha(p+\mu-2)+1 \le 0$. (iii)

$$\int_0^{\infty} \frac{d\eta}{g^{\beta}(\eta)} \to \infty \quad (5)$$

The impulsive points meet the following criteria. (iv) There is a constant M such that

$$\left|\sum_{i=1}^{r} x_i\right| < N, \quad n=1,2,\ldots\ldots \quad (6)$$

If a solution to (1) is $w(v)$, then

$$\lim_{v\to\infty} \sup \frac{|w(v)|}{v} < \infty \quad (7)$$

Proof. From (4), we get that

$$|w(v)| \le |w_0| + \frac{\lambda}{\Gamma p}\int_0^v (v-w)^{p-1}|e(w)|dw + \left|\sum_{i=1}^{r} y_i\right| + \frac{1}{\Gamma p}\int_0^v (v-w)^{p-1}|f(w,z(w))|dw$$
$$+\sum_{i=1}^{n} b_i(w)g_i\left(z\left(w-\tau_i\right)\right) \quad \text{for } u\in(v_r,v_{r+1}]$$

Then the condition (II) is applied, we have

$$|w(v)| \le \left|w_0 + \frac{\lambda}{\Gamma p}\int_0^v (v-w)^{p-1}\right| e(w)\left| dw + \left|\sum_{i=1}^{r} y_i\right| + \frac{1}{\Gamma p}\int_0^u (v-w)^{p-1}w^{\mu-1}h(w)g\left(\frac{|z(w)|}{w}\right)\right.$$
$$dw + \sum_{i=1}^{n} b_i(w)g_i\left(z\left(w-\tau_i\right)\right) \text{ for vin}\left(v_r,v_{r+1}\right].$$

We obtain $\frac{\lambda}{u}\int_0^v (v-w)^{p-1}|e(w)|dw \le c$ for all $u \ge b$

where c is constant from 1. Consider $C(r) = |w_0| + \left|\sum_{i=1}^{r} y_i\right| + \frac{c}{\Gamma p}$ We have

$$|z(u)| \le C(r)v + \frac{1}{\Gamma p}\int_0^v (v-w)^{p-1}w^{\mu-1}h(w)g\left(\frac{|z(w)|}{w}\right)dw + \sum_{i=1}^{n} b_i(w)$$
$$g_i\left(z\left(w-\tau_i\right)\right)dw$$
$$\le C(r)v + \frac{1}{\Gamma p}(u)\int_0^v (v-w)^{p-2}w^{\mu-1}h(w)g\left(\frac{|z(w)|}{w}\right)dw + u\sum_{i=1}^{n} b_i(w)g_i\left(z\left(w-\tau_i\right)\right)dw$$
$$\text{for} \quad v\in\left(v_r,v_{r+1}\right].$$

Thus, the inequality results

$$\frac{|w(v)|}{v} \leq C(r) + \frac{1}{\Gamma p}\int_0^v (v-w)^{p-2} w^{\mu-1} h(w) g\left(\frac{|z(w)|}{w}\right) dw + \sum_{i=1}^n b_i(w) g_i\left(z\left(w-\tau_i\right)\right) dw \quad \text{for } v \in \left(v_r, v_{r+1}\right]. \tag{8}$$

Assume we denote the right side of inequality as $x(v)$. We then obtain the following inequality:

$$\frac{|w|}{v} \leq x(v,r), \quad \text{vin}\left(v_r, v_{r+1}\right] \tag{9}$$

Given that the function g is non-decreasing, inequality can be expressed as:

$$g\left(\frac{|w|}{v}\right) \leq g(x(v,r)), \quad v \in \left(v_r, v_{r+1}\right]$$

Thus we obtain from the definition of $x(v,r)$

$$x(v,r) \leq 1 + C(r) + \frac{1}{\Gamma p}\int_0^v (v-w)^{\delta-1} w^{\mu-1} h(w) g(x(w,r)) dw + \sum_{i=1}^n b_i(w) g_i\left(z\left(w-\tau_i\right)\right) dw \quad \text{for } v \in \left(v_r, v_{r+1}\right] \tag{10}$$

where $0 < \delta = p-1 < 1$. Applying Lemma (6) and Holders inequality, we get

$$\int_0^v (v-w)^{\delta-1} w^{\mu-1} h(w) g(x(w,r)) dw + \sum_{i=1}^n \int_0^v b_i(w) g_i\left(z\left(w-\tau_i\right)\right) dw \leq \left(\int_0^v (v-w)^{\alpha(\delta-1)} w^{\alpha(\mu-1)} dw\right)^{\frac{1}{\alpha}} \left(\int_0^v h^\beta(w) g^\beta(x(w,r)) dw\right)^{\frac{1}{\beta}} + \sum_{i=1}^n \left(\int_0^v b_i^\alpha(w) dw\right)^{1/\alpha} \left(\int_0^v g_i^\beta\left(z\left(w-\tau_i\right)\right) dw\right)^{1/\beta} \leq \left(Bu^\theta\right)^{\frac{1}{\alpha}} \left(\int_0^v h^\beta(w) g^\beta(x(w,r)) dw\right)^{\frac{1}{\beta}} + \left(Au^\theta\right)^{1/\alpha} \sum_{i=1}^n \left(\int_0^v g_i^\beta\left(z\left(w-\tau_i\right)\right) dw\right)^{1/\beta}, \text{ for } \quad v \in \left(v_r, v_{r+1}\right]$$

where $\theta = \alpha(p+\mu-2)+1 \leq 0$, and $B := B(\alpha(\mu-1)+1, \alpha(\delta-1)+1)$. Utilizing the fact that $u > w > m$ and $\theta \leq 0$

$$\int_0^t (v-w)^{\delta-1} w^{\mu-1} h(w) g(x(w,r)) dw + \sum_{i=1}^{n} \int_0^v b_i(w) g_i(z(w-\tau_i)) dw \leq (B)^{1/\alpha}$$
$$\left(\int_0^v w^{\theta\beta/\alpha} h^\beta(w)\right.$$
$$\left. g^\beta(z(w,r)) dw\right)^{1/\beta} + (A)^{1/\alpha} \sum_{i=1}^{n} \left(\int_0^v g_i^\beta (z(w-\tau_i)) dw\right)^{1/\beta}, \quad \text{for} \quad \text{vin}\,(v_r, v_{r+1}].$$

Using the elementary inequality

$$(y+z)^\beta \leq 2^{\beta-1}\left(y^\beta + z^\beta\right), \quad y, \quad z \geq 0, \quad \beta > 1 \quad (11)$$

For $u \in (u_r, u_{r+1}]$, we infer that

$$x^\beta(v,r) \leq 2^{\beta-1}\left[(1+C(r))^\beta + \left(B^{1/\alpha}\frac{1}{\Gamma p}\right)^\beta \int_m^v w^{\theta\beta/\alpha} h^\beta(w) g^\beta(x(w,r)) dw \right.$$
$$\left. + \left(A^{1/\alpha}\frac{1}{\Gamma p}\right)^\beta \sum_{i=1}^{n}\left(\int_0^v g_i^\beta (z(w-\tau_i)) dw\right)\right].$$

If we denote $P_1(r) = 2^{\beta-1}(1+C(r))^\beta$, $Q_1 = 2^{\beta-1}\left(B^{1/\alpha}\frac{1}{\Gamma\alpha}\right), R_1 = \left(A^{1/\alpha}\frac{1}{\Gamma p}\right)$ then

$$x^\beta(v,r) \leq P_1(r) + Q_1 \textstyle\int_0^v w^{\theta\beta/\alpha} h^\beta(w) g^\beta(x(w,r)) dw + R_1 \sum_{i=1}^n \left(\int_0^v g_i^\beta (z(w\tau_i)) dw\right.$$
$$\text{for} \quad v \in (v_r, v_{r+1}].$$

Describe

$$w\eta = g^\beta(\eta)$$
$$G(\xi) + \int_{x_r}^{\xi} \frac{d\eta}{w(\eta)}, \quad x_r = x\left(v_r^+, r\right) \quad (12)$$

Since $G(x(v,r)) = \int_{x_r}^{x(v,r)} \frac{d\eta}{g^\beta(\eta)}$, condition (iii) suggests that $\lim_{x(v,r)\to\infty} G(x(v,r)) = \infty$, we can obtain using the Bihari Lemma,

$$x^{\beta}(v,r) \le R(r) := G^{-1}\left(G(p_1(r)) + Q_1 \int_0^v w^{\theta\beta/\alpha} h^{\beta}(w)dw + R_1 \sum_{i=1}^{n}\left(\int_0^v g_i^{\beta}(z(w-\tau_i))\,dw\right)\right)$$
$$\text{for} \quad v \in (v_r, v_{r+1}], \quad r = 1,2,\ldots.$$

Due to the boundedness of $P_1(r)$ and condition (iv). Thus , we can conclude $R(r)$, $r = 1,2,\ldots.$ is bounded by (12). Then

$$x^{\beta}(v,r) \le R = \sup_{r\ge 1} R(r), \quad v > v_1, \quad u = 1,2,\ldots..$$

We drive that $x(v,r) \le R^{1/\beta}$, and using equation (8), we get

$$\frac{|z(v)|}{v} \le R^{1/\beta}, \quad v \ge v_1.$$

We conclude that

$$\lim_{v\to\infty} \sup \frac{|z(v)|}{v} < \infty$$

This completes the proof.

Theorem 1 *Let $\alpha,\beta,\mu,p,q,$ and θ be given as in previous Theorem 2.1 , with the conditions (5) to (10) satisfied. If any constant $d_1 \in (M + z_0, 1 + M + z_0)$,*

$$\lim_{v\to\infty} inf[d_1.u + \int_0^v (v-w)^{p-1} e(w)dw] = -\infty \quad (13)$$

$$\lim_{v\to\infty} sup[d_1.v + \int_0^v (v-w)^{p-1} e(w)dw] = \infty \quad (14)$$

subsequently, (1) oscillates.

III. CONCLUSION

In this study, we explore impulsive fractional differential equations (1) that encompass Hilfer derivatives, accounting for both initial and impulsive conditions. By utilizing these conditions, we establish sufficient criteria for the oscillation of solutions to equations (1). The oscillatory nature of the equation is then confirmed through the application of the inequality principle and Bihari Lemma. Furthermore, an illustrative example is presented to highlight the main results.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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REFERENCES

[1] A. Hermosillo-Arteaga, M. P. Romo, and R. Magana-del Toro, "Response spectra generation using a fractional differential model," Soil Dynamics and Earthquake Engineering **115**, 719–729 (2018).

[2] Y. Jiang, B. Xia, X. Zhao, T. Nguyen, C. Mi, and R. A. de Callafon, "Data-based fractional differential models for nonlinear dynamic modeling of a lithium-ion battery," Energy **135**, 171–181 (2017).

[3] L. Feng and S. Sun, "Oscillation theorems for three classes of conformable fractional differential equations," Advances in Difference Equations **2019**, 1–30 (2019).

[4] Y. Bolat, "On the oscillation of fractional-order delay differential equations with constant coefficients," Communications in Nonlinear Science and Numerical Simulation **19**, 3988–3993 (2014).

[5] Y. Z. Wang, Z. L. Han, Z. H. A. O. Ping, and S. R. Sun, "Oscillation theorems for fractional neutral differential equations," Hacettepe Journal of Mathematics and Statistics **44**, 1477–1488 (2015).

[6] Y. Zhou, B. Ahmad, and A. Alsaedi, "Existence of nonoscillatory solutions for fractional neutral differential equations," Applied Mathematics Letters **72**, 70–74 (2017).

[7] Q. Ma, J. Pecaric, and J. Zhang, "Integral inequalities of systems and the estimate for solutions of certain nonlinear two-dimensional fractional differential systems," Computers and Mathematics with Applications **61**, 3258–3267 (2011).

[8] A. Ortega, J. J. Rosales, J. M. Cruz-Duarte, *et al.*, "Fractional model of the dielectric dispersion," Optik-International Journal for Light and Electron Optics **180**, 754–759 (2019).

[9] A. Raheem and M. Maqbul, "Oscillation criteria for impulsive partial fractional differential equations," Computers and Mathematics with Applications **73**, 1781–1788 (2017).

[10]I. Stamova, "Global stability of impulsive fractional differential equations," Applied Mathematics and Computation **237**, 605–612 (2014).

[11]S. R. Grace, R. P. Agarwal, J. Y. Wong, and A. Zafer, "On the oscillation of fractional differential equations," Fractional Calculus and Applied Analysis **15**, 222–231 (2012).

[12]J. Wang, X. Li, and W. Wei, "On the natural solution of an impulsive fractional differential equation of order $q \in (1,2)$," Communications in Nonlinear Science and Numerical Simulation **17**, 4384–4394 (2012).

[13]P. Prakash, S. Harikrishnan, J. Nieto, and K. Hoon, "Oscillation of a time fractional partial differential equation," Electronic Journal of Qualitative Theory of Differential Equations **2014**, 1–10 (2014).

[14]D. X. Chen, "Oscillatory behavior of a class of fractional differential equations with damping," UPB Scientific Bulletin, Series A **75**, 107–118 (2013).

[15]D. X. Chen, "Oscillation criteria of fractional differential equations," Advances in Difference Equations **2012**, 1–10 (2012).

[16]D. X. Chen, "Oscillation criteria of fractional differential equations," Advances in Difference Equations **2012**, 1–10 (2012).

[17]W. N. Li, "Oscillation of solutions for certain fractional partial differential equations," Advances in Difference Equations **2016**, 1–8 (2016).

[18]W. N. Li, "On the forced oscillation of certain fractional partial differential equations," Applied Mathematics Letters **50**, 5–9 (2015).

[19]W. N. Li, "Forced oscillation criteria for a class of fractional partial differential equations with damping term," Mathematical Problems in Engineering **2015**, Article ID 393624 (2015).

[20]M. M. El-Borai, "Some probability densities and fundamental solutions of fractional evolution equations," Chaos, Solitons and Fractals **14**, 433–440 (2002).

[21]Y. Zhou and F. Jiao, "Existence of mild solutions for fractional neutral evolution equations," Computers and Mathematics with Applications **59**, 1063–1077 (2010).

[22]S. Liu, X. Wu, X. F. Zhou, and W. Jiang, "Asymptotical stability of riemann–liouville fractional nonlinear systems," Nonlinear Dynamics **86**, 65–71 (2016).

[23]S. Liu, X. F. Zhou, X. Li, and W. Jiang, "Asymptotical stability of riemann–liouville fractional singular systems with multiple time-varying delays," Applied Mathematics Letters **65**, 32–39 (2017).

[24]K. Li, J. Peng, and J. Jia, "Cauchy problems for fractional differential equations with riemann–liouville fractional derivatives," Journal of Functional Analysis **263**, 476–510 (2012).

[25]R. P. Agarwal and A. Özbekler, "Lyapunov type inequalities for mixed nonlinear riemann–liouville fractional differential equations with a forcing term," Journal of Computational and Applied Mathematics **314**, 69–78 (2017).

[26]R. P. Agarwal, V. Lupulescu, D. O'Regan, and G. ur Rahman, "Fractional calculus and fractional differential equations in nonreflexive banach spaces," Communications in Nonlinear Science and Numerical Simulation **20**, 59–73 (2015).

[27]H. Liu, Y. Pan, S. Li, and Y. Chen, "Adaptive fuzzy backstepping control of fractional-order nonlinear systems," IEEE Transactions on Systems, Man, and Cybernetics: Systems **47**, 2209–2217 (2017).

[28]H. Liu, S. Li, G. Li, and H. Wang, "Adaptive controller design for a class of uncertain fractional-order nonlinear systems: an adaptive fuzzy approach," International Journal of Fuzzy Systems **20**, 366–379 (2018).

[29]R. Hilfer, Y. Luchko, and . Tomovski, "Operational method for the solution of fractional differential equations with generalized riemann-liouville fractional derivatives," Fractional Calculus and Applied Analysis **12**, 289–318 (2009).

[30]X. Ding and J. J. Nieto, "Controllability and optimality of linear time-invariant neutral control systems with different fractional orders," Acta Mathematica Scientia **35**, 1003–1013 (2015).