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Novel Insights into Oscillation of Impulsive Fractional Differential Equations with Caputo Derivative

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ABSTRACT: In this paper, we explore the oscillation of impulsive Caputo fractional differential equations. Conditions for both asymptotic and oscillatory outcomes are established through the application of the inequality principle and Bihari Lemma. An example is given to explain the results of all problems. This is the first time to study the oscillation of impulsive fractional differential equation with Caputo Derivative.

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I. INTRODUCTION

Fractional differential equations are generalizations of classical differential equations of integer order and can find applications in many fields of science and engineering It has different research areas on mechanical and electrical properties of real materials,as well as in rheological theory and other physical problems,see[1–3]. For articles on the oscillation of fractional differential equations,readers can refer to literatures[4–10].

Fractional differential equation has been the focus of many studies due to their frequent appearance in many applications in physics,biology, control theory,control processing, engineering and has attracted more and more scholars.The oscillation of impulsive fractional differential equation as a new research field and the new interesting results have already been obtained.Due to the intensive development about the theory of impulsive differential equations and fractional calculus and their widely applications in diverse fields,impulsive fractional differential equations have become a new hot topic. Very recently,more and more researchers show great interest in the field of impulsive problems for fractional differential equations,see[7,11–13].

In 2016, Jessada Tariboon and Sotiris K.Ntouyas[14]investigated oscillation results for the solutions of impulsive fractional differential equation with conformable derivative of the form,

$$\begin{cases} {}_{t_k}D^{\alpha}\left(p(t)\left[{}_{t_k}D^{\alpha}x(t)+r(t)x(t)\right]\right)+q(t)x(t)=0, \quad t>t_0, \quad t\neq t_k, \quad \alpha\in(1,2), \\ x\left(t_k^+\right)=a_k x\left(t_k^-\right), {}_{t_k}D^{\alpha}x\left(t_k^+\right)=b_{k\,t_{k-1}}D^{\alpha}x\left(t_k^-\right), k=1,2,\ldots\ldots \end{cases}$$

They obtained some new oscillatory results by using the equivalence transformation and the associated Riccati techniques.

The definition of conformable derivative is only related to the limit form and is similar to form of integer derivative.Therefore,the methods for oscillation of integer differential equation can be applied to conformable derivative only through a simple transformation.There are still some gaps between conformable derivative and classical

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fractional derivative .
In 2017, A.Raheem, Md.Maqbul[15] considered the oscillatory behavior of solutions on the differential equation with Riemann-Liouville fractional derivative, for $t \neq t_j$

$$D^{\beta}_{+,t}u(x,t)+a(t)D^{\beta-1}_{+,t}u(x,t)=b(t)\triangle u(x,t)+\sum_{k=1}^{m}c_k(t)\triangle u(x,t-\tau_k)-F(x,t)$$

under the impulsive condition,

$$D^{\beta-1}_{+,t}u\left(x,t_j^+\right)-D^{\beta-1}_{+,t}u\left(x,t_j^-\right)=\sigma(x,t_j)D^{\beta-1}_{+,T}u(x,t_j),\quad j=1,2,\ldots..\quad (x,t)\in\Omega.\mathbb{R}_+$$

With two kind of boundary condition

$$\frac{\partial u(x,t)}{\partial N}+f(x,t)u(x,t)=0,\quad (u,t)\in\partial\Omega\cdot\mathbb{R}_+,\quad t\neq t_j$$

and

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\cdot\mathbb{R}_+,\quad t\neq t_j$$

where $a,b,c_k \in PC[\mathbb{R}_+,\mathbb{R}_+]$, forcing term $F \in PC[\bar{\Omega}.\mathbb{R}_+,\mathbb{R}_+]$,$f \in PC[\partial\Omega,\mathbb{R}^+]$,and PC denotes the class of functions which are piecewise continuous functions in t with discontinuities of first kind only at $t=t_j$, j=1,2,....and left continuous at $t=t_j$, $\beta\in(1,2)$ is a constant , $\triangle$ is the Laplacian operator in $\mathbb{R}^n$,Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega,\Omega=\Omega\cup\partial\Omega$, N is the unit out normal vector to $\partial\Omega$.
In 2019, Mouffak Benchohra , Samira Hamani and Yong Zhou [3] dealt with the existence of oscillatory and non oscillatory solutions for the following class of initial value problems for impulsive fractional differential with Caputo-Hadamard derivative inclusion,

$$\begin{cases} H_cD^{\alpha}_{t_k}y(t)\in F(t,y(t)),t\in J=(t_k,t_k+1), \\ y\left(t_K^+\right)=I_k\left(y\left(t_k^-\right)\right),k=1,2,\ldots\ldots. \\ y(1)=y_*. \end{cases}$$

By using the concept of upper and lower solutions and the fixed point, theorem,the authors,obtained the existence theorems of oscillatory and non-oscillatory solutions of the above equation.
Motivated by the above papers, we consider the oscillatory behavior os solutions of following fractional impulsive differential equation

$$\begin{cases} {}^cD^{\beta}_{a}x(t)=\lambda e(t)+f(t,x(t))+\sum_{i=1}^{n}b_i(t)g_i(x(t-\tau_i)) \\ t\in[o,T] \\ \Delta x(t_k)=y_k,\Delta x'(t_k)=\overline{y_k},k=1,2,\ldots\ldots. \\ x(t)=\phi(t),x'(t)=\bar{\phi}(t),t\in[v,o] \\ \alpha\beta<1 \end{cases} \tag{1}$$

II. MAIN RESULTS

We are now in a position to state and prove our main results.

Theorem 1 *Suppose that* $1<\alpha<2,\quad p>1,\quad \gamma>$

$0,\ \ p(\alpha-2)+1>0,\ \ p(\gamma-1)+1>0,\ \ q=\dfrac{p}{p-1}$, *and the function* $e(t): J \to \mathbb{R}$ *is continuous such that*

$$\frac{1}{t}\int_a^t (t-s)^{\alpha-1}|e(s)|ds \ \ \text{is bounded for all } t \geq a \quad (2)$$

and the function $f(t,x)$ *satisfies the following conditions.*
(i) $f(t,x)$ *is continuous in* $D=(t,x): t\in J,\ \ x\in\mathbb{R}$.
(ii) There are continuous nonnegetive functions $g,\ h:\mathbb{R}^+ := [a,\infty)\to\mathbb{R}^+$, *g is nondecreasing and let* $0<\gamma\leq 3-\alpha-1/p$ *such that*

$$|f(t,x)| \leq t^{\gamma-1}\ h(t)g(\frac{|x|}{t}), \qquad t>a, \qquad (t,x)\in D, \ (3)$$

and

$$\int_a^\infty s^{\theta q/p}h^q(s)ds < \infty, \quad (4)$$

where $\theta := p(\alpha+\gamma-3)+1 \leq 0$.
(iii)

$$\int_a^\infty \frac{d\eta}{g^q(\eta)} \to \infty. \quad (5)$$

The impulsive points meet the following condition.
(iv) There is a constant M such that

$$|\sum_{i=1}^k \bar{y}_i| < M, \qquad |\sum_{i=1}^k y_i| < M, \qquad k=1,2,........ \quad (6)$$

If $x(t)$ *is a solution of (1), then*

$$\lim_{t\to\infty}\sup\frac{|x(t)|}{t} < \infty \quad (7)$$

Proof. We obtain from (1) that

$$|x(t)| \leq \phi(t) + \frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|e(s)|ds + \left|\sum_{i=1}^k \overline{y_i}\right| t + \left|\sum_{i=1}^k y_i\right| + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} |f(s,x(s))|ds + \frac{1}{\Gamma(\beta)}\sum_{i=1}^n b_i(s)\int_0^t (t-s)^{\beta-1} g_i(x(s-\tau i))ds \quad t\in(t_k,t_{k+1}].$$

By applying condition (2) we get,

$$|x(t)| \leq \phi(t) + \tfrac{\lambda}{\Gamma(\beta)}\textstyle\int_0^t (t-s)^{\beta-1}|e(s)|ds + \left|\sum_{i=1}^k \bar{y}_i\right| t + \sum_{i=1}^k y_i\Big| + \tfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} s^{\gamma-1}h(s)g\left(\tfrac{|x(s)|}{s}\right)ds + \tfrac{1}{\Gamma(\beta)}\sum_{i=1}^n b_i(s)\int_0^t (t-s)^{\beta-1} g_i(x(s-\tau i))ds \quad t\in(t_k,t_{k+1}]$$

From (1), we obtain $\frac{1}{t}\int_0^t (t-s)^{\beta-1}|e(s)|ds \leq c$ for all $t\geq a$, where d is a constant.

Let $C(k) = \phi(t) + |\sum_{i=1}^k \bar{y}_i| + |\sum_{i=1}^k yi| + \dfrac{\lambda c}{\Gamma(\beta)} + \dfrac{1}{\Gamma\beta}\sum_{i=1}^n b_i(s)$. We have

$$|x(t)| \leq C(k)t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g\left(\frac{|x(s)|}{s}\right) ds + \int_0^t (t-s)^{\beta-1} g_i(x(s-\tau i)) ds \quad t \in (t_k, t_{k+1}]$$
$$\leq C(k)t + \frac{1}{\Gamma(\beta)} t \int_0^t (t-s)^{\beta-2} s^{\gamma-1} h(s) g\left(\frac{|x(s)|}{s}\right) ds + \int_0^t (t-s)^{\beta-1} g_i(x(s-\tau i)) ds \quad t \in (t_k, t_{k+1}].$$

This yields the inequality.

$$\frac{|x(t)|}{t} \leq C(k) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-2} s^{\gamma-1} h(s) g\left(\frac{|x(s)|}{s}\right) ds + \frac{1}{t} \int_0^t (t-s)^{\beta-1} g_i(x(s-\tau i)) ds \quad t \in (t_k, t_{k+1}]. \tag{8}$$

If we denote that $z(t)$ is the right side of the inequality (7). We obtain the inequality

$$\frac{|x(t)|}{t} \leq z(t,k), \quad t \in (t_k, t_{k+1}] \tag{9}$$

Since the function g is non decreasing, the inequality (8) yields

$$g(\frac{|x(t)|}{t}) \leq g(z(t,k)), \qquad t \in (t_k, t_{k+1}]$$

$$g\left(\frac{|x(t)|}{t}\right) \leq g(z(t,k)), \quad t \in (t_k, t_{k+1}]$$

and from a definition $z(t,k)$,we get

$$z(t,k) \leq 1 + C(k) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s,k)) ds + \frac{1}{t} \int_0^t (t-s)^{\beta-1} g_i(x(s-\tau i)) ds \quad t \in (t_k, t_{k+1}] \tag{10}$$

Where $0 < \alpha = \beta - 1 < 1$.

Applying Holders inequality and Lemma we obtain

$$\int_0^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s,k)) ds + \int_0^t (t-s)^{\beta-1} g_i(x(s-\tau i)) ds$$
$$\leq \left(\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds\right)^{1/p} \left(\int_0^t h^q(s) g^q(z(s,k)) ds\right)^{1/q} + \left(\int_0^t (t-s)^{p(\beta-1)} ds\right)^{1/p}$$
$$\left(\int_0^t g_i^q(x(s-\tau i)) ds\right)^{1/q}$$
$$\leq \left(\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds\right)^{1/p} \left(\int_0^t h^q(s) g^q(z(s,k)) ds\right)^{1/q}$$
$$\leq \left(Bt^{\theta}\right)^{1/p} \left(\int_0^t h^q(s) g^q(z(s,k)) ds\right)^{1/q} + \left(B' t^{\theta}\right)^{1/p} \left(\int_0^t g_i^q(x(s-\tau i)) ds\right)^{1/q}$$
$$, t \in (t_k, t_{k+1}]$$

where $B := B(p(\gamma-1)+1 < p(\beta-1)+1)$, $\theta = p(\alpha+\gamma-3)+1 \leq 0$. Using the fact that $\theta \leq 0$ and $t > s \geq 0$, we have

$$\int_0^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s,k)) ds + \int_0^t (t-s)^{\beta-1} g_i(x(s-\tau i)) ds$$
$$\leq B^{1/p} \left(\int_0^t s^{\theta q/p} h^q(s) g^q(z(s,k)) ds\right)^{1/q} \left(B' t^{\theta}\right)^{1/p} \left(\int_0^t g_i^q(x(s-\tau i)) ds\right)^{1/q} \quad , t \in (t_k, t_{k+1}] \tag{11}$$

Using (10) and the elementary inequality

$$(x+y)^q \leq 2^{q-1}(x^q + y^q), \qquad x, \quad y \geq 0, \qquad q > 1$$

.
For $t \in (t_k, t_{k+1}]$,we obtain from 9 that

$$z^q(t,k) \leq 2^{q-1}(1+C(k))^q + \left(B^{1/p} \tfrac{1}{\Gamma(\beta)}\right)^q \int_0^t s^{\theta q/p} h^q(s) g^q(z(s,k)) ds\Big) + \left(\tfrac{1}{t} B'^{1/p}\right)^q \int_0^t g_i^{q\theta/p}(x(s-\tau i)) ds$$

If we denote

$$p_1(k) = 2^{q-1}\left[(1+C(k))^q\right], \quad Q_1 = 2^{q-1}\left(\left(B^{1/p} \frac{1}{\Gamma(\beta)}\right)^q, R_1 = \left(\frac{1}{t} B'^{1/p}\right)^q,\right.$$

then

$$z^{q}(t,k)\leq p_{1}(k)+Q_{1}\int_{0}^{t}s^{\theta q/p}h^{q}(s)g^{q}(z(s,k))ds+R_{1}\int_{0}^{t}g_{i}^{q\theta/p}(x(s-\tau i))ds,\quad t\in(t_{k},t_{k+1}].$$

Denote

$$\omega(\eta)=g^{q}(\eta)$$
$$G(\xi)=\int_{z_{k}}^{\xi}\frac{d\eta}{\omega(\eta)},\quad z_{k}=z\left(t_{k}^{+},k\right) \tag{12}$$

Since $G(z(t,k))=\int_{z_{k}}^{z^{(t,k)}}\frac{d\eta}{g^{q}(\eta)}$, condition (iii) implies that $\lim_{z(t,k)\to\infty} G(z(t,k))=\infty$, then by the Bihari Lemma[16], we get

$$z^{q}(t,k)\leq K(k):=G^{-1}\Big(G(p_{1}(k))+Q_{1}\int_{0}^{t}s^{\theta q/p}h^{q}(s)ds$$
$$+R_{1}\int_{0}^{t}g_{i}^{q\theta/p}(x(s-\tau i))ds,\quad t\in(t_{k},t_{k+1}],\quad k=1,2,\ldots\ldots$$

Because of condition (iv) and boundedness of $P_1(k)$. Hence from (ii) we conclude $K(k)$, $k=1,2,....$ is bounded.Then

$$z^{q}(t,k)\leq K=\sup_{k\geq 1}K(k),\quad t>t_{1},\quad k=1,2,\ldots\ldots$$

We obtain that $z(t,k)\leq K^{1/q}$, and from (8), we have

$$\frac{|x(t)|}{t}\leq K^{1/q},\quad t\geq t_{1}.$$

We conclude that

$$\lim_{t\to\infty}\frac{|x(t)|}{t}<\infty.$$

This completes the proof.

Theorem 2 *Let the constants β, p, q, γ and θ be defined as is in Theorem 2.1, conditions (1)-(5) hold.If for any constant $\bar{d}\in(MT(\beta)+\bar{\phi_{\star}}\Gamma(\beta),\quad 1+M\Gamma(\beta)+\bar{\phi_{\star}}\Gamma(\beta))$,*

$$\lim_{t\to\infty} inf\left[\bar{d}t+\int_{0}^{t}(t-s)^{\beta-1}e(s)ds\right]=-\infty \tag{13}$$

or

$$\lim_{t\to\infty} sup\left[\bar{d}t+\int_{0}^{t}(t-s)^{\beta-1}e(s)ds\right]=\infty \tag{14}$$

then (1) is oscillatory.

III. CONCLUSION

In the final analysis, because of significant measure to their vital functions in describing an assortment of biological, physiological, and technological occurrences, the investigation of differential equations with fractions and regressive parabolic problems has grown dramatically. Mastering complex structures, especially the rigidity and electrical features of substance and the exploration of rheological actions, have been found to be particularly helpful for employing fractional differential equations, which extends the conventional view of mathematical equations to non-integer levels.These mathematical formulas are growing becoming increasingly significant in categories like physics, biology, control theory, and architecture since they could symbolize processes containing memory and inherited features.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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